

Besov Spaces of Self-affine Lattice Tilings and Pointwise regularity

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Abstract

We investigate Besov spaces of self-affine tilings of \mathbb{R}^n and discuss various characterizations of those Besov spaces. We see what is a finite set of functions which generates the Besov spaces from a view of multiresolution approximation on self-affine lattice tilings of \mathbb{R}^n . Using this result we give a generalization of already known characterizations of Besov spaces given by wavelet expansion and we apply to study the pointwise Hölder space. Furthermore we give descriptions of scaling exponents measured by Besov spaces, and estimations of a pointwise Hölder exponent to compute the pointwise scaling exponent of several oscillatory functions.

1 Introduction

There are many ways to characterize Besov spaces. Among them in the discrete version are regular wavelet expansion, Littlewood-Paley decomposition, polynomial approximation, spline approximation, mean oscillation, and difference operator (See [9], [13] and [15]). In this paper we give these characterizations in context of self-affine lattice tilings of \mathbb{R}^n and we apply to study these pointwise versions. In particular we see to give most of these characterizations in a framework of multiresolution approximation on self-affine lattice tilings of \mathbb{R}^n . We also give conditions of finitely many functions which generate the Besov spaces of self-affine lattice tilings of \mathbb{R}^n in a view of multiresolution approximation scheme. This result is a generalization of characterizations of Besov spaces given by regular wavelet functions and by spline functions. (See [3], [12] and [15]). Moreover we apply to give descriptions of scaling exponents by characterizations of the Besov space, and we also consider a pointwise Hölder exponent of oscillatory functions given by a multiresolution approximation series in self-affine lattice tilings of \mathbb{R}^n .

The plan of sections in our paper is as follows:

In the second section we introduce self-affine lattice tilings of \mathbb{R}^n which arise in many contexts, particularly, in fractal geometry and in construction of wavelet bases. See [14] for a survey on related topics. We define Besov spaces of self-affine lattice tilings, and give its characterizations and its pointwise versions.

In the third section we consider a multiresolution analysis $\{V_l\}$ generated by finitely many functions associated with a self-affine lattice tiling. We give properties of Besov space norms defined by approximation errors associated with $\{V_l\}$.

In the fourth section we give some conditions of finitely many functions which characterize the Besov space by multiresolution approximation on self-affine lattice tilings of \mathbb{R}^n .

We apply this result to give a generalization of characterizations of Besov spaces given by regular wavelet functions and by spline functions, and we also give characterizations of the pointwise Hölder space by multiresolution approximation.

In the fifth section we give descriptions of scaling exponents of global and pointwise regularity by characterizations of the Besov space. We give properties of a pointwise Hölder exponent for a multiresolution approximation series in self-affine lattice tilings and apply to compute a pointwise Hölder exponent of several oscillatory functions.

We use C to denote a positive constant different in each occasion. But it will depend on the parameter appearing in each problem. The same notations C are not necessarily the same on any two occurrences.

2 Self-affine lattice tilings and Besov spaces

Let Γ be a lattice in \mathbb{R}^n , that is, Γ is an image of the integer lattice \mathbb{Z}^n under some nonsingular linear transformation and let M be a dilation matrix, that is, all eigenvalues of M have absolute values greater than one and M preserves the lattice Γ : $M\Gamma \subset \Gamma$. This implies that $|\det M| = m$ is a positive integer greater than one and m is the order of the quotient space $\Gamma/M\Gamma$. We say that a compact set T generates a self-affine tiling $\{T + \gamma\}_{\gamma \in \Gamma}$ if

$$\begin{aligned} \cup_{\gamma \in \Gamma} (T + \gamma) &= \mathbb{R}^n \text{ disjoint a.e.} \\ \cup_{\gamma \in \Gamma_0} (T + \gamma) &= MT \text{ disjoint a.e.} \end{aligned} \tag{1}$$

where Γ_0 is a finite subset of Γ consisting of representatives for disjoint cosets in $\Gamma/M\Gamma$. The set Γ_0 is called a set of digits and the compact set T is called a self-affine tile. The self-affine tile T has nonempty interior T° . For convenience we restrict our attention to the case $\Gamma = \mathbb{Z}^n$. In this case the dilation matrix M has integer entries.

For $1 \leq p \leq \infty$, let $\mathcal{L}^p = \mathcal{L}^p(\mathbb{R}^n)$ be the linear space of all functions ϕ for which

$$|\phi|_p = \left(\int_{\mathbb{R}^n} \left(\sum_{\nu \in \mathbb{Z}^n} |\phi(x - \nu)| \right)^p dx \right)^{1/p} < \infty. \tag{2}$$

with the usual modification for $p = \infty$. Clearly, $\mathcal{L}^p \subset L^p(\mathbb{R}^n)$ and $\mathcal{L}^\infty \subset \mathcal{L}^p \subset \mathcal{L}^q \subset \mathcal{L}^1 = L^1(\mathbb{R}^n)$ for $1 \leq q \leq p \leq \infty$. If $\phi \in L^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$) is compactly supported, then $\phi \in \mathcal{L}^p$. Furthermore, we observe that if there are constants $C > 0$ and $\delta > 0$ such that $|\phi(x)| \leq C(1 + |x|)^{-n-\delta}$ for all $x \in \mathbb{R}^n$ then $\phi \in \mathcal{L}^\infty$.

A finite subset $\Phi = \{\phi_1, \dots, \phi_N\}$ of \mathcal{L}^∞ is said to have L^p -stable shifts ($1 \leq p \leq \infty$), if there are constants $C_1 > 0$ and $C_2 > 0$ such that for any sequences $c_j \in l^p(\mathbb{Z}^n)$ ($j = 1, \dots, N$),

$$C_1 \sum_{j=1}^N \|c_j\|_{l^p} \leq \left\| \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} c_j(\nu) \phi_j(x - \nu) \right\|_p \leq C_2 \sum_{j=1}^N \|c_j\|_{l^p}.$$

From now those equivalences shall be described as

$$\sum_{j=1}^N \|c_j\|_{l^p} \sim \left\| \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} c_j(\nu) \phi_j(x - \nu) \right\|_p.$$

Theorem A ([6]). For a finite subset $\Phi = \{\phi_1, \dots, \phi_N\}$ of \mathcal{L}^∞ , we have following equivalent conditions:

- (i) Φ has L^2 -stable shifts,
- (ii) Φ has L^p -stable shifts for $1 \leq p \leq \infty$,
- (iii) there is a set of functions $\tilde{\Phi} = \{\tilde{\phi}_1, \dots, \tilde{\phi}_N\}$ in \mathcal{L}^∞ , dual to Φ in the sense that

$$\int \phi_j(x - \mu) \tilde{\phi}_k(x - \nu) dx = \delta_{\mu\nu} \delta_{jk}, \quad j, k = 1, \dots, N, \quad \mu, \nu \in \mathbb{Z}^n,$$

where δ is the Kronecker's symbol.

Let $\Pi = \{T + \nu\}_{\nu \in \mathbb{Z}^n}$ be a self-affine lattice tiling of \mathbb{R}^n with a dilation matrix M . For a nonnegative integer k , we denote the function p_α with $|\alpha| \leq k$, $\alpha \in \mathbb{Z}_+^n$, where \mathbb{Z}_+ is the set of all nonnegative integers, given by

$$\begin{aligned} p_\alpha(x) &= x^\alpha, \quad x \in T^o \\ p_\alpha(x) &= 0 \quad \text{otherwise.} \end{aligned} \tag{3}$$

Since $\Phi = \{p_\alpha\}_{|\alpha| \leq k}$ of \mathcal{L}^∞ has L^2 -stable shifts, there is a set of functions $\tilde{\Phi} = \{\tilde{p}_\alpha\}_{|\alpha| \leq k}$ dual to Φ .

Let Q_0 be a translate of the tile T containing the origin as an interior point and let $p'_\alpha, \tilde{p}'_\alpha$ be corresponding translates of $p_\alpha, \tilde{p}_\alpha$ respectively. For $Q_l(x_0) = M^{-l}Q_0 + x_0$, we write

$$p_\alpha^{Q_l(x_0)}(x) = m^{l/2} p'_\alpha(M^l(x - x_0)), \quad \tilde{p}_\alpha^{Q_l(x_0)}(x) = m^{l/2} \tilde{p}'_\alpha(M^l(x - x_0))$$

$$P_{Q_l(x_0)} f(x) = \sum_{|\alpha| \leq k} \langle f, \tilde{p}_\alpha^{Q_l(x_0)} \rangle p_\alpha^{Q_l(x_0)}(x). \tag{4}$$

We define

$$\text{osc}_p^k f(x, l) = \inf_{P \in \mathbb{P}^k} \left(\frac{1}{|Q_l(x)|} \int_{Q_l(x)} |f(y) - P(y)|^p dy \right)^{1/p} \tag{5}$$

and

$$\Omega_p^k f(x, l) = \left(\frac{1}{|Q_l(x)|} \int_{Q_l(x)} |f(y) - P_{Q_l(x)} f(y)|^p dy \right)^{1/p}$$

where $Q_l(x) = M^{-l}Q_0 + x$ and $P_{Q_l(x)} f$ is given in (4), and $|Q_l(x)|$ is the volume element of $Q_l(x)$, and \mathbb{P}^k is the linear space of all polynomials of degree no greater than k on \mathbb{R}^n .

Definition. Let λ_0 be the least value of absolute values of eigenvalues of the dilation matrix M . Given $s > 0$, k a nonnegative integer with $k + 1 > s$ and $1 \leq p, q \leq \infty$. A function f is said to belong to the Besov space $B_{pq}^s(M)$ if

$$\|f\|_{B_{pq}^s(M)} = \|f\|_p + \left(\sum_{l=0}^{\infty} (\lambda_0^l \|\text{osc}_p^k f(\cdot, l)\|_p)^q \right)^{1/q} < \infty. \tag{6}$$

with the usual modification for $q = \infty$. We note that the above definition is independent of the choice of nonnegative integers k with $k + 1 > s$ and osc_p^k in the definition can be

replaced by osc_1^k . We can see $W_{k+1}^p(\mathbb{R}^n) \subset B_{pq}^s(M)$ if $s < k+1$. When the dilation matrix M is λ_0 -times of the identity Id with $\lambda_0 > 1$, the above Besov space coincides the usual Besov space on \mathbb{R}^n .

Remark 1. We have the embedding theorem : $B_{p\xi}^\beta(M) \subset B_{p\eta}^\alpha(M)$ for $\beta > \alpha > 0$, $1 \leq \xi, \eta \leq \infty$ and $1 \leq p \leq \infty$, and $B_{p\xi}^\alpha(M) \subset B_{p\eta}^\alpha(M)$ for $\alpha > 0$, $1 \leq \xi \leq \eta \leq \infty$ and $1 \leq p \leq \infty$.

Let $\Delta_u f$ denote the difference operator $\Delta_u f(x) = f(x+u) - f(x)$. Let us choose positive constants r and d such that

$$\{u \in \mathbb{R}^n : |u| < r\} \subset Q_0 \subset \{u \in \mathbb{R}^n : |u| < dr\}. \quad (7)$$

Theorem 1 . *Given $s > 0$, a nonnegative integer k with $k+1 > s$ and $1 \leq p, q \leq \infty$, we have equivalent ones of the Besov space norm given in (6), if one of them exists, with the usual modification for $q = \infty$,*

$$\begin{aligned} & \|f\|_{B_{pq}^s(M)} \\ & \sim \|f\|_p + \left(\sum_{l=0}^{\infty} (\lambda_0^{ls} \|\Omega_p^k f(\cdot, l)\|_p)^q \right)^{1/q} \equiv |||f|||_1, \\ & \sim \|f\|_p + \left(\sum_{l=0}^{\infty} (\lambda_0^{ls} \sup_{(k+1)|M^l u| < r/2} \|\Delta_u^{k+1} f\|_p)^q \right)^{1/q} \equiv |||f|||_2. \end{aligned}$$

Proof. Since $\text{osc}_p^k f(x, l) \sim \Omega_p^k f(x, l)$ from [2], the equivalence of $\|f\|_{B_{pq}^s(M)}$ and $|||f|||_1$ is obvious.

We shall prove for any f such that $|||f|||_2 < \infty$,

$$\|\Omega_p^k f(\cdot, l)\|_p \leq C \left(\sup_{(k+1)|M^l u| < r/2} \|\Delta_u^{k+1} f\|_p + \lambda_0^{-l(k+1)} \|f\|_p \right).$$

We choose a function χ in $C_c^\infty(\mathbb{R}^n)$ such that $\int |\chi(u)| du = 1$ and $\text{supp} \chi \subset \{u \in \mathbb{R}^n : |u| < r/2(k+1)\}$. We write $\chi_l(u) = m^l \chi(M^l u)$, $h_l(x) = \int (f(x) - \Delta_u^{k+1} f(x)) \chi_l(u) du$. Then we have

$$\begin{aligned} \Omega_p^k f(x, l) & \leq \left(\frac{1}{|Q_l(x)|} \int_{Q_l(x)} |f(y) - h_l(y)|^p dy \right)^{1/p} + \left(\frac{1}{|Q_l(x)|} \int_{Q_l(x)} |h_l(y) - P_{Q_l(x)} h_l(y)|^p dy \right)^{1/p} \\ & + \left(\frac{1}{|Q_l(x)|} \int_{Q_l(x)} |P_{Q_l(x)} h_l(y) - P_{Q_l(x)} f(y)|^p dy \right)^{1/p} \\ & \leq C \left(\left(\frac{1}{|Q_l(x)|} \int_{Q_l(x)} |f(y) - h_l(y)|^p dy \right)^{1/p} + \Omega_p^k h_l(x, l) \right) \equiv C(I_1(x) + I_2(x)). \end{aligned}$$

We have :

$$\|I_1\|_p \leq \|(m^l \int_{M^{-l} Q_0} (\int |\Delta_u^{k+1} f(\cdot + y)| |\chi_l(u)| du)^p dy)^{1/p}\|_p \leq C \sup_{(k+1)|M^l u| < r/2} \|\Delta_u^{k+1} f\|_p.$$

Let q_z be the k -th Taylor polynomial of h_l about $z \in \mathbb{R}^n$. To estimate I_2 , we use

$$|\partial^\beta h_l(x)| \leq C \sum_{e=1}^{k+1} \int_{|u| < r/2(k+1)} |f(x - eM^{-l}u)| du \leq C \int_{|u| < r/2} |f(x - M^{-l}u)| du.$$

Hence we get an estimate:

$$\begin{aligned}
\text{osc}_p^k h_l(x, l) &\leq C(m^l \int_{Q_l(x)} |h_l(y) - q_x(y)|^p dy)^{1/p} \\
&\leq C(m^l \int_{Q_l(x)} |\int_0^1 \sum_{|\beta|=k+1} \frac{k+1}{\beta!} \partial^\beta h_l(x + t(y-x))(1-t)^k (y-x)^\beta dt|^p dy)^{1/p} \\
&\leq C(m^l \int_{M^{-l}Q_0} (\int_0^1 \int_{|u|<r/2} |f(x+ty - M^{-l}u)| |y|^{k+1} dt du)^p dy)^{1/p}.
\end{aligned}$$

Hence, we get an estimate :

$$||I_2||_p \leq C(m^l \int_{M^{-l}Q_0} |y|^{(k+1)p} ||f||_p^p dy)^{1/p} \leq C||f||_p \lambda_0^{-l(k+1)}.$$

Now we combine the estimates of I_1 and I_2 to write

$$||\Omega_p^k f(\cdot, l)||_p \leq C(||I_1||_p + ||I_2||_p) \leq C(\sup_{(k+1)|M^l u|<r/2} ||\Delta_u^{k+1} f||_p + \lambda_0^{-l(k+1)} ||f||_p).$$

This shows $|||f|||_1 \leq C|||f|||_2$.

We shall show the converse. We can show from [5] for $|M^l u| < r/2(k+1)$,

$$|\Delta_u^{k+1} f(x)| = |\Delta_u^{k+1} (f - P_{Q_l(x)} f)(x)| \leq C \sum_{e=0}^{k+1} \sum_{j=l}^{\infty} \Omega_p^k f(x + eu, j).$$

Hence we have

$$\sup_{(k+1)|M^l u|<r/2} ||\Delta_u^{k+1} f||_p \leq C \sum_{j=l}^{\infty} ||\Omega_p^k f(\cdot, j)||_p$$

and we get the estimate

$$|||f|||_2 \leq C|||f|||_1.$$

This completes the proof of Theorem 1.

If $0 < s < k+1$ for a nonnegative integer k and $1 \leq p, q \leq \infty$, then for $x \in \mathbb{R}^n$, a function $f \in T_{pq}^s(x)$ means that

$$(\sum_{l=0}^{\infty} (\lambda_0^{ls} \text{osc}_p^k f(x, l))^q)^{1/q} < \infty$$

with the usual modification for $q = \infty$.

Remark 2. We have the embedding theorem : $T_{p\xi}^\beta(x) \subset T_{p\eta}^\alpha(x)$ for $\beta > \alpha > 0$, $1 \leq \xi, \eta \leq \infty$ and $1 \leq p \leq \infty$, and $T_{p\eta}^\alpha(x) \subset T_{p\xi}^\alpha(x)$, $T_{\xi q}^\alpha(x) \subset T_{\eta q}^\alpha(x)$ for $\alpha > 0$, $1 \leq \eta \leq \xi \leq \infty$ and $1 \leq p, q \leq \infty$.

We have a poinwise version of Theorem 1, which is proved by the same way as the proof of Theorem 1.

Corollary. Given $s > 0$, a nonnegative integer k with $k + 1 > s$ and $1 \leq p, q \leq \infty$. Then for $x \in \mathbb{R}^n$ following properties of a bounded function f are equivalent, with the usual modification for $q = \infty$,

- (i) $f \in T_{pq}^s(x)$,
- (ii) $(\sum_{l=0}^{\infty} (\lambda_0^{ls} \Omega_p^k f(x, l))^q)^{1/q} < \infty$,
- (iii) $(\sum_{l=0}^{\infty} (\lambda_0^{ls} \sup_{(k+1)|M^l u| < r/2} (\frac{1}{|Q_l(x)|} \int_{Q_l(x)} |\Delta_u^{k+1} f(y)|^p dy)^{1/p})^q)^{1/q} < \infty$.

We will define the Littlewood-Paley decomposition. Let us $\lambda_0 > 1$ and φ a function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ with the following properties: $\text{supp } \hat{\varphi} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 1\}$ and $\hat{\varphi}(\xi) = 1$ on $\{\xi \in \mathbb{R}^n : |\xi| \leq \lambda_0^{-1}\}$. Let $\psi(x) = \lambda_0^n \varphi(\lambda_0 x) - \varphi(x)$. Let $\varphi_l(x) = \lambda_0^{ln} \varphi(\lambda_0^l x)$, $S_l f = f * \varphi_l$, $\psi_l(x) = \lambda_0^{ln} \psi(\lambda_0^l x)$ and $f_l = f * \psi_l$ for $l = 0, 1, 2, \dots$. Then for $f \in \mathcal{S}'$ we have Littlewood-Paley decomposition:

$$f = \varphi * f + \sum_{l=0}^{\infty} \psi_l * f \equiv S_0 f + \sum_{l=0}^{\infty} f_l. \quad (8)$$

Theorem B ([13]). Suppose that a dilation matrix is of the form $M = \lambda_0 Id$ with $\lambda_0 > 1$. Let $1 \leq p, q \leq \infty$ and $s > 0$. Then we have equivalence of norms if one of them exit, for Littlewood-Paley decomposition given in (8), with the usual modification $q = \infty$:

$$\begin{aligned} & \text{(i)} \quad \|f\|_{B_{pq}^s(M)}, \\ & \sim \text{(ii)} \quad \|f\|_p + (\sum_{l=0}^{\infty} (\lambda_0^{ls} \|f - S_l f\|_p)^q)^{1/q}, \\ & \sim \text{(iii)} \quad \|S_0 f\|_p + (\sum_{l=0}^{\infty} (\lambda_0^{ls} \|f_l\|_p)^q)^{1/q}. \end{aligned}$$

We write $T_{\infty\infty}^s(x) = C^s(x)$. The following statement is a pointwise version of Theorem B.

Proposition 1 . Suppose that a dilation matrix is of the form $M = \lambda_0 Id$ with $\lambda_0 > 1$. Let $s > 0$. Then for $x \in \mathbb{R}^n$, following properties of a bounded function f for Littlewood-Paley decomposition given in (8) are equivalent:

- (i) $f \in C^s(x)$,
- (ii) $|f(y) - S_l f(y)| \leq C(\lambda_0^{-l} + |x - y|^s)$ for all $l \geq 0$.

Proof. We will prove that (i) implies (ii). Let us $k + 1 > s$. By [1: Theorem 2] we have

$$f(x) - S_l f(x) = \int \Delta_u^{k+1} f(y) \varphi_l(u) du - \sum_{e=1}^k \binom{k}{e} (-1)^e f * \psi_l^e(y)$$

where $\varphi^e(y) = e^{-n}\varphi(e^{-1}y)$ and $\psi^e = \varphi^e - \varphi^{e+1}$. From [1: Lemma 2] there exist functions $\psi^{e_1}, \dots, \psi^{e_n}$ in $\mathcal{S}(\mathbb{R}^n)$ such that

$$\psi_l^e(x) = \sum_{i=1}^n \Delta_{c\lambda_0^{-l}e_i}^{k+1} \psi_l^{ei}(x)$$

where $c = \lambda_0^{-1}e\pi$ and e_1, \dots, e_n are the canonical basis vectors in \mathbb{R}^n to write

$$f * \psi_l^e = \sum_{i=1}^n f * (\Delta_{c\lambda_0^{-l}e_i}^{k+1} \psi_l^{ei}) = \sum_{i=1}^n (\Delta_{c\lambda_0^{-l}e_i}^{k+1} f) * \psi_l^{ei}.$$

We get from the corollary of Theorem 1

$$\begin{aligned} |f * \psi_l^e(y)| &\leq C \sum_{i=1}^n \int |\Delta_{c\lambda_0^{-l}e_i}^{k+1} f(y-z) \psi_l^{ei}(z)| dz \\ &\leq C \sum_{i=1}^n \int (|x-y+z| + c\lambda_0^{-l})^s |\psi_l^{ei}(z)| dz \leq C \sum_{i=1}^n \int (|x-y| + |z| + \lambda_0^{-l})^s |\psi_l^{ei}(z)| dz \\ &\leq C \sum_{i=1}^n (\lambda_0^{-l} + |x-y|)^s \int (1+|z|)^s |\psi_l^{ei}(z)| dz \leq C(\lambda_0^{-l} + |x-y|)^s. \end{aligned}$$

Furthermore we see

$$\begin{aligned} \int |\Delta_u^{k+1} f(y) \varphi_l(u)| du &\leq C \int (|x-y| + |u|)^s |\varphi_l(u)| du \\ &\leq C(\lambda_0^{-l} + |x-y|)^s \int (1+|u|)^s |\varphi_l(u)| du \leq C(\lambda_0^{-l} + |x-y|)^s \end{aligned}$$

These give the estimate

$$\begin{aligned} |f(y) - S_l f(y)| &\leq C \int |\Delta_u^{k+1} f(y) \varphi_l(u)| du + C \sum_{e=1}^k |f * \psi_l^e(y)| \leq C(\lambda_0^{-l} + |x-y|)^s. \end{aligned}$$

This completes the implication of (ii) from (i).

We will show that (ii) implies (i). We choose a positive integer l_1 such that $\lambda_0^{l_1} > d+1$ where d is given in (7).

We have

$$|\Delta_u^{k+1} f(y)| \leq |\Delta_u^{k+1} (f - S_l f)(y)| + |\Delta_u^{k+1} S_l f(y)|$$

We will give an estimate of $|\Delta_u^{k+1} (f - S_l f)(y)|$ to see

$$\begin{aligned} &\sup_{Q_{l+l_1}(x)} \sup_{(k+1)|M^{l+l_1}u| < r/2} |\Delta_u^{k+1} (f - S_l f)(y)| \\ &\leq C \sup_{Q_{l+l_1}(x)} \sup_{(k+1)|M^{l+l_1}u| < r/2} \sum_{e=0}^{k+1} |(f - S_l f)(y + eu)| \leq C \sup_{Q_l(x)} |(f - S_l f)(y)|. \end{aligned}$$

We will give an estimate of $|\Delta_u^{k+1} S_l f(y)|$ to write

$$|\Delta_u^{k+1} S_l f(y)| \leq C \sum_{|\beta|=k+1} |u|^{k+1} \int_0^1 \cdots \int_0^1 |\partial^\beta S_l f(y + (\theta_1 + \cdots + \theta_{k+1})u)| d\theta_1 \cdots d\theta_{k+1}.$$

We have by (ii)

$$|f_l(y)| \leq |f(y) - S_{l+1}f(y)| + |f(y) - S_l f(y)| \leq C(\lambda_0^{-l} + |x - y|)^s.$$

Hence, to estimate $|\partial^\beta S_l f(y)|$ with $|\beta| = k + 1$ we use, by Bernstein's inequality,

$$\begin{aligned} |\partial^\beta S_l f(y)| &\leq \sum_{j=0}^{l-1} |\partial^\beta f_j(y)| + |\partial^\beta S_0 f(y)| \\ &\leq C \left(\sum_{j=0}^{l-1} \lambda_0^{(k+1)j} (\lambda_0^{-j} + |x - y|)^s + \|f\|_\infty \right) \leq C \lambda_0^{(k+1)l} (\lambda_0^{-l} + |x - y|)^s. \end{aligned}$$

From those we have

$$\sup_{(k+1)|M^l u| < r/2} |\Delta_u^{k+1} S_l f(y)| \leq C \sup_{(k+1)|M^l u| < r/2} |u|^{k+1} \lambda_0^{(k+1)l} (\lambda_0^{-l} + |x - y| + |u|)^s \leq C(\lambda_0^{-l} + |x - y|)^s.$$

Hence we get an estimate

$$\begin{aligned} &\sup_{Q_{l+l_1}(x)} \sup_{(k+1)|M^{l+l_1} u| < r/2} |\Delta_u^{k+1} f(y)| \\ &\leq C \sup_{Q_l(x)} |(f - S_l f)(y)| + \sup_{Q_l(x)} \sup_{(k+1)|M^l u| < r/2} |\Delta_u^{k+1} S_l f(y)| \leq C \lambda_0^{-ls}. \end{aligned}$$

These complete the proof of Proposition 1 by the corollary of Theorem 1.

A following corollary may be proved by the same way as in the proof of Proposition 1.

Corollary. *Suppose that a dilation matrix $M = \lambda_0 Id$. Let f be a bounded function. If $f \in C^s(x)$, then it holds*

$$(iii) \quad |f_l(y)| \leq C(\lambda_0^{-l} + |x - y|)^s \quad \text{for all } l \geq 0.$$

Conversely, if it holds for $s > s' > 0$,

$$(iii)' \quad |f_l(y)| \leq C \lambda_0^{-ls} (1 + \lambda_0^l |x - y|)^{s'} \quad \text{for all } l \geq 0,$$

then $f \in C^s(x)$.

3 Multiresolution approximation

Let Π denote a self-affine lattice tiling $\{T + \nu\}_{\nu \in \mathbb{Z}^n}$ with a dilation matrix M . For an integer l and a finite subset $\Phi = \{\phi_1, \dots, \phi_N\}$ of \mathcal{L}^∞ with L^2 -stable shifts, we define operators $P_l f$ given by

$$P_l f(x) = \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} m^l \langle f, \tilde{\phi}_j(M^l \cdot - \nu) \rangle \phi_j(M^l x - \nu) \quad (9)$$

where $\langle f, \tilde{\phi}_j(M^l \cdot - \nu) \rangle = \int f(y) \tilde{\phi}_j(M^l y - \nu) dy$ and $\tilde{\Phi} = \{\tilde{\phi}_1, \dots, \tilde{\phi}_N\}$ is dual to Φ in Theorem A.

Let $V_0^p = \{\sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} a_j(\nu) \phi_j(x - \nu) : a_j \in l^p(\mathbb{Z}^n)\}$ and let $V_l^p = \{f(M^l x) : f \in V_0^p\}$. Then for $1 \leq p \leq \infty$, the operator P_l is a bounded projection operator of $L^p(\mathbb{R}^n)$ onto V_l^p ($1 \leq p \leq \infty$) in the sense that $P_l f = f$ for any $f \in V_l^p$. We say $\Phi = \{\phi_1, \dots, \phi_N\}$ of \mathcal{L}^∞ is M -refinable if there exist sequences $c_{jk} \in l^1(\mathbb{Z}^n)$ ($1 \leq j, k \leq N$) such that

$$\phi_j(x) = \sum_{k=1}^N \sum_{\nu \in \mathbb{Z}^n} c_{jk}(\nu) \phi_k(Mx - \nu), \quad x \in \mathbb{R}^n, \quad j = 1, \dots, N.$$

A following theorem implies that $\{V_l^p\}$ is a multiresolution analysis in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

Theorem C ([6] and [16]). *If a finite subset Φ of \mathcal{L}^∞ is M -refinable and has L^2 -stable shifts, then the sequence of sets $\{V_l^p\}$ ($1 \leq p \leq \infty$) satisfies following properties:*

- (i) $f \in V_0^p \Leftrightarrow f(x - \nu) \in V_0^p$ for all $\nu \in \mathbb{Z}^n$,
- (ii) $f \in V_l^p \Leftrightarrow f(Mx) \in V_{l+1}^p$,
- (iii) $\dots \subset V_l^p \subset V_{l+1}^p \subset \dots$,
- (iv) $\cap_{l \in \mathbb{Z}} V_l^p = \{0\}$ ($1 \leq p < \infty$),
- (v) $\cup_{l=0}^\infty V_l^p$ is dense in $L^p(\mathbb{R}^n)$ ($1 \leq p < \infty$).

Given a function f in $L^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$), $\sigma_l^p(f)$ denotes the error of L^p -approximation from V_l^p in $L^p(\mathbb{R}^n)$:

$$\sigma_l^p(f) = \inf\{\|f - S\|_p : S \in V_l^p\}. \quad (10)$$

Clearly we have the following equivalence:

$$\sigma_l^p(f) \sim \|f - P_l f\|_p, \quad f \in L^p(\mathbb{R}^n) \quad (1 \leq p \leq \infty).$$

Given $s > 0$, $\lambda > 1$ and $1 \leq p, q \leq \infty$. A function f is said to belong to $B_{pq}^{s, \lambda}(\Phi)$ if

$$\|f\|_{B_{pq}^{s, \lambda}(\Phi)} = \|f\|_p + \left(\sum_{l=0}^\infty (\lambda^{ls} \sigma_l^p(f))^q\right)^{1/q} < \infty \quad (11)$$

with the usual modification when $q = \infty$.

Let

$$R_l f = P_{l+1} f - P_l f, \quad l = 0, 1, \dots \quad (12)$$

We put

$$P_0 f(x) = \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} a_{j0}(\nu) \phi_j(x - \nu), \quad R_l f(x) = \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} a_{j(l+1)}(\nu) \phi_j(M^{l+1} x - \nu). \quad (13)$$

Since Φ has stable shifts, we have

$$\|P_0 f\|_p \sim \sum_{j=1}^N \|a_{j0}\|_{l^p}, \quad \|R_l f\|_p \sim m^{-(l+1)/p} \sum_{j=1}^N \|a_{j(l+1)}\|_{l^p}, \quad l = 0, 1, \dots \quad (14)$$

Then for $f \in B_{pq}^{s, \lambda}(\Phi)$ we have

$$f(x) = P_0 f(x) + \sum_{l=0}^\infty R_l f(x) \equiv \sum_{j=1}^N \sum_{l=0}^\infty \sum_{\nu \in \mathbb{Z}^n} a_{jl}(\nu) \phi_j(M^l x - \nu).$$

Moreover from [15, Theorem 5.10] there exists an associated set of wavelets $\{\psi_j^\epsilon\}_{j=1,\dots,N}^{\epsilon=1,\dots,m-1}$, that is, $\{\psi_j^\epsilon(x-\nu)\}_{j=1,\dots,N,\nu\in\mathbb{Z}^n}^{\epsilon=1,\dots,m-1}$ is an orthonormal basis in $W_0 = V_1^2 \ominus V_0^2$ in $L^2(\mathbb{R}^n)$, whose wavelet expansion of a function $f \in L^2(\mathbb{R}^n)$ is given by

$$f(x) = \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} a_{j0}(\nu) \phi_j(x - \nu) + \sum_{j=1}^N \sum_{\epsilon=1}^{m-1} \sum_{l=0}^{\infty} \sum_{\nu \in \mathbb{Z}^n} b_{jl}^\epsilon(\nu) m^{l/2} \psi_j^\epsilon(M^l x - \nu) \quad (15)$$

where

$$a_{j0}(\nu) = \langle f(y), \tilde{\phi}_j(y - \nu) \rangle, \quad b_{jl}^\epsilon(\nu) = \langle f(y), m^{l/2} \psi_j^\epsilon(M^l y - \nu) \rangle. \quad (16)$$

Then we have

$$P_0 f(x) = \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} a_{j0}(\nu) \phi_j(x - \nu),$$

$$R_l f(x) = \sum_{j=1}^N \sum_{\epsilon=1}^{m-1} \sum_{\nu \in \mathbb{Z}^n} b_{jl}^\epsilon(\nu) m^{l/2} \psi_j^\epsilon(M^l x - \nu), l = 0, 1, \dots$$

When $m > (n+1)/2$, there exist $\psi_j^\epsilon \in \mathcal{L}^\infty$ and

$$\|R_l f\|_p \sim m^{l(1/2-1/p)} \sum_{j=1}^N \sum_{\epsilon=1}^{m-1} \|b_{jl}^\epsilon\|_{l^p} \quad (1 \leq p \leq \infty).$$

Theorem 2 . Assume that a finite subset $\Phi = \{\phi_1, \dots, \phi_N\}$ of \mathcal{L}^∞ is M -refinable and has L^2 -stable shifts. Given $\lambda > 1$ and $\alpha > 0$, there are equivalences of the norm $\|f\|_{B_{pq}^{\alpha,\lambda}(\Phi)}$ given in (11), if one of them exists, for any $1 \leq p, q \leq \infty$, with the usual modification for $q = \infty$:

- (i) $\|f\|_p + (\sum_{l=0}^{\infty} (\lambda^{l\alpha} \|f - P_l f\|_p)^q)^{1/q}$,
- (ii) $\|P_0 f\|_p + (\sum_{l=0}^{\infty} (\lambda^{l\alpha} \|R_l f\|_p)^q)^{1/q}$,
- (iii) $(\sum_{l=0}^{\infty} (\lambda^{l\alpha} m^{-l/p} \sum_{j=1}^N \|a_{jl}\|_{l^p})^q)^{1/q}$, where $\{a_{jl}\}$ are given in (13).
- (iv) $\inf(\sum_{l=0}^{\infty} (\lambda^{l\alpha} m^{-l/p} \sum_{j=1}^N \|c_{jl}\|_{l^p})^q)^{1/q}$ where the infimum is taken over all admissible L^p -convergent representations

$$f(x) = \sum_{j=1}^N \sum_{l=0}^{\infty} \sum_{\nu \in \mathbb{Z}^n} c_{jl}(\nu) \phi_j(M^l x - \nu),$$

- (v) $\sum_{j=1}^N \|a_{j0}\|_{l^p} + (\sum_{l=0}^{\infty} (\lambda^{l\alpha} m^{l(1/2-1/p)} \sum_{j=1}^N \sum_{\epsilon=1}^{m-1} \|b_{jl}^\epsilon\|_{l^p})^q)^{1/q}$ when $m > (n+1)/2$, where $\{a_{j0}\}$ and $\{b_{jl}^\epsilon\}$ are given in (16).

Proof. Those equivalences of (i), (ii), (iii), (v) and (11) can be proved from easy routine using Hardy's inequality. We will prove only the equivalence of (iv) and (11). We prove that

$$\|f\|_p + (\sum_{l=0}^{\infty} (\lambda^{l\alpha} \sigma_l^p(f))^q)^{1/q} \leq C (\sum_{l=0}^{\infty} (\lambda^{l\alpha} m^{-l/p} \sum_{j=1}^N \|c_{jl}\|_{l^p})^q)^{1/q} < \infty$$

when $f(x) = \sum_{j=1}^N \sum_{l=0}^{\infty} \sum_{\nu \in \mathbb{Z}^n} c_{jl}(\nu) \phi_j(M^l x - \nu)$ is L^p -convergent.

We put $f(x) = \sum_{l=0}^{\infty} F_l(x)$, $F_l(x) = \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} c_{jl}(\nu) \phi_j(M^l x - \nu)$. Then we see $F_l \in V_l^p$ and $\|F_l\|_p \sim m^{-l/p} \sum_{j=1}^N \|c_{jl}\|_{l^p}$.

Hence we have

$$\begin{aligned} \sigma_{l_0}^p(f) &= \sigma_{l_0}^p\left(\sum_{l=0}^{\infty} F_l\right) \\ &\leq \sum_{l=0}^{\infty} \sigma_{l_0}^p(F_l) = \sum_{l=0}^{l_0} \sigma_{l_0}^p(F_l) + \sum_{l=l_0+1}^{\infty} \sigma_{l_0}^p(F_l) = \sum_{l=l_0+1}^{\infty} \sigma_{l_0}^p(F_l) \leq \sum_{l=l_0+1}^{\infty} \|F_l\|_p. \end{aligned}$$

From Hardy's inequality this implies

$$\begin{aligned} \|f\|_p + \left(\sum_{l_0=0}^{\infty} (\lambda^{l_0\alpha} \sigma_{l_0}^p(f))^q\right)^{1/q} &\leq \sum_{l=0}^{\infty} \|F_l\|_p + \left(\sum_{l_0=0}^{\infty} (\lambda^{l_0\alpha} \sum_{l=l_0+1}^{\infty} \|F_l\|_p^q)\right)^{1/q} \\ &\leq C \left(\sum_{l=0}^{\infty} (\lambda^{l\alpha} \|F_l\|_p^q)\right)^{1/q} \leq C \left(\sum_{l=0}^{\infty} (\lambda^{l\alpha} m^{-l/p} \sum_{j=1}^N \|c_{jl}\|_{l^p}^q)\right)^{1/q}. \end{aligned}$$

From (iii), we have

$$\|f\|_{B_{pq}^{\alpha,\lambda}(\Phi)} \sim \left(\sum_{l=0}^{\infty} (\lambda^{l\alpha} m^{-l/p} \sum_{j=1}^N \|a_{jl}\|_p^q)\right)^{1/q}.$$

This implies the equivalence of (iv) and (11).

Proposition 2 . *Given $k+1 > s > 0$. Assume that $\Phi = \{\phi_1, \dots, \phi_N\}$ of \mathcal{L}^{∞} is M -refinable and has L^2 -stable shifts. Then we have for any $1 \leq p, q \leq \infty$,*

$$B_{pq}^{s,\lambda_0}(\Phi) \subset B_{pq}^s(M)$$

provided that there exists a positive number s_0 with $s_0 > s$ such that $\sup_{l \geq 0} \lambda^{ls_0} |\text{osc}_p^k \phi_j(\cdot, l)|_p < \infty$ for all $j = 1, \dots, N$, where the norm $|\cdot|_p$ and osc_p^k are given in (2) and (5) respectively, and λ_0 is the least value of absolute values of eigenvalues of M .

Proof. We shall prove for any $f \in B_{pq}^{s,\lambda_0}(\Phi)$,

$$\left(\sum_{l=0}^{\infty} (\lambda_0^{ls} \tilde{\sigma}_l^p(f))^q\right)^{1/q} \leq C(\|f\|_p + \left(\sum_{l=0}^{\infty} (\lambda_0^{ls} \sigma_l^p(f))^q\right)^{1/q})$$

where σ_l^p is the errors of L^p -approximation given in (10) associated with Φ and $\tilde{\sigma}_l^p(f) = \|\text{osc}_p^k f(\cdot, l)\|_p$. Since $\sigma_l^p(f) \rightarrow 0$ as $l \rightarrow \infty$ ($1 \leq p \leq \infty$), we have an L^p -convergent series

$$f(x) = P_0 f(x) + \sum_{l=0}^{\infty} R_l f(x) \equiv \sum_{j=1}^N \sum_{l=0}^{\infty} \sum_{\nu \in \mathbb{Z}^n} a_{jl}(\nu) \phi_j(M^l x - \nu)$$

where $P_0 f(x) = \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} a_{j0}(\nu) \phi_j(x - \nu)$ and $R_l f(x) = \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} a_{j(l+1)}(\nu) \phi_j(M^{l+1} x - \nu)$ are given in (13).

Then we have

$$\begin{aligned}\tilde{\sigma}_{l_0}^p(f) &= \tilde{\sigma}_{l_0}^p(P_0f + \sum_{l=0}^{\infty} R_l f) \\ &\leq \tilde{\sigma}_{l_0}^p(P_0f) + \sum_{l=0}^{\infty} \tilde{\sigma}_{l_0}^p(R_l f) \equiv I_0 + \sum_{l=0}^{\infty} I'_l.\end{aligned}$$

We shall give an estimate of I_0 . By (14) we have

$$\begin{aligned}I_0 &\leq C \sum_{j=1}^N \left\| \sum_{\nu \in \mathbb{Z}^n} |a_{j0}(\nu)| \text{osc}_p^k \phi_j(x - \nu, l_0) \right\|_p \\ &\leq C \sum_{j=1}^N \|a_{j0}\|_p |\text{osc}_p^k \phi_j(\cdot, l_0)|_p \leq C \|P_0f\|_p \sup_j |\text{osc}_p^k \phi_j(\cdot, l_0)|_p.\end{aligned}$$

If $l < l_0$, then we see by (14) that

$$\begin{aligned}I'_l &\leq C m^{-(l+1)/p} \sum_{j=1}^N \left\| \sum_{\nu} |a_{j(l+1)}(\nu)| \text{osc}_p^k \phi_j(x - \nu, l_0 - l - 1) \right\|_p \\ &\leq C \sum_{j=1}^N m^{-(l+1)/p} \|a_{j(l+1)}\|_p |\text{osc}_p^k \phi_j(\cdot, l_0 - l - 1)|_p \leq C \|R_l f\|_p \sup_j |\text{osc}_p^k \phi_j(\cdot, l_0 - l - 1)|_p.\end{aligned}$$

If $l \geq l_0$, then we have by the definition,

$$I'_l \leq \|R_l f\|_p.$$

Hence the above estimates of I_0 and I'_l , imply that

$$\begin{aligned}\tilde{\sigma}_{l_0}^p(f) &\leq C \|P_0f\|_p \sup_j |\text{osc}_p^k \phi_j(\cdot, l_0)|_p \\ &\quad + C \sum_{l=0}^{l_0-1} \|R_l f\|_p \sup_j |\text{osc}_p^k \phi_j(\cdot, l_0 - l - 1)|_p + \sum_{l=l_0}^{\infty} \|R_l f\|_p\end{aligned}$$

and from Hardy's inequality and Theorem 2,

$$\begin{aligned}& \left(\sum_{l_0=0}^{\infty} (\lambda_0^{l_0 s} \tilde{\sigma}_{l_0}^p(f))^q \right)^{1/q} \\ &\leq C \left(\sum_{l_0=0}^{\infty} (\|P_0f\|_p \lambda_0^{l_0 s} \sup_j |\text{osc}_p^k \phi_j(\cdot, l_0)|_p)^q \right)^{1/q} \\ &\quad + C \left(\sum_{l_0=0}^{\infty} \left(\sum_{l=0}^{l_0-1} \|R_l f\|_p \lambda_0^{l_0 s} \sup_j |\text{osc}_p^k \phi_j(\cdot, l_0 - l - 1)|_p \right)^q \right)^{1/q} + \left(\sum_{l_0=0}^{\infty} \left(\sum_{l=l_0}^{\infty} \|R_l f\|_p \lambda_0^{l_0 s} \right)^q \right)^{1/q} \\ &\leq C \sup_{j, l \geq 0} \lambda_0^{l s_0} |\text{osc}_p^k \phi_j(\cdot, l)|_p \left\{ \|P_0f\|_p \left(\sum_{l_0=0}^{\infty} \lambda_0^{-l_0(s_0-s)q} \right)^{1/q} \right. \\ &\quad \left. + \left(\sum_{l_0=0}^{\infty} \left(\sum_{l=0}^{l_0-1} \|R_l f\|_p \lambda_0^{l_0 s} \lambda_0^{-(l_0-l-1)s_0} \right)^q \right)^{1/q} \right\} + \left(\sum_{l_0=0}^{\infty} \left(\sum_{l=l_0}^{\infty} \|R_l f\|_p \lambda_0^{l_0 s} \right)^q \right)^{1/q} \\ &\leq C \sup_{j, l \geq 0} \lambda_0^{l s_0} |\text{osc}_p^k \phi_j(\cdot, l)|_p (\|P_0f\|_p + \left(\sum_{l=0}^{\infty} (\|R_l f\|_p \lambda_0^{l s})^q \right)^{1/q}) \\ &\leq C \sup_{j, l \geq 0} \lambda_0^{l s_0} |\text{osc}_p^k \phi_j(\cdot, l)|_p (\|f\|_p + \left(\sum_{l=0}^{\infty} (\sigma_l^p(f) \lambda_0^{l s})^q \right)^{1/q}).\end{aligned}$$

This completes the proof of Proposition 2.

A following corollary can be proved by the same way in the proof of Proposition 2.

Corollary. *Given $\lambda > 1$ and $s > 0$. Assume that $\Phi = \{\phi_1, \dots, \phi_N\}$ and $\Phi' = \{\phi'_1, \dots, \phi'_L\}$ of \mathcal{L}^∞ are M -refinable and have L^2 -stable shifts. Then we have for any $1 \leq p, q \leq \infty$,*

$$B_{pq}^{s,\lambda}(\Phi') \subset B_{pq}^{s,\lambda}(\Phi)$$

provided that there exists a positive number s_0 with $s_0 > s$ such that $\sup_{l \geq 0} \lambda^{ls_0} |\phi'_j - P_l \phi'_j|_p < \infty$ for all $j = 1, \dots, L$, where the operator P_l is given in (9) associated with Φ .

For a positive integer k and $1 \leq p \leq \infty$, $\mathcal{L}_k^p = \mathcal{L}_k^p(\mathbb{R}^n)$ is denoted to be the space of all functions f such that $f(x)(1 + |x|)^k \in \mathcal{L}^p$. If $\phi \in L^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$) is compactly supported, then $\phi \in \mathcal{L}_k^p$. Furthermore, we observe that if there are constants $C > 0$ and $\delta > k$ such that $|\phi(x)| \leq C(1 + |x|)^{-n-\delta}$ for all $x \in \mathbb{R}^n$ then $\phi \in \mathcal{L}_k^\infty$.

For a finite subset Φ of \mathcal{L}_k^∞ , the domain of the operator P_l given in (9), can be extended to include the linear space \mathbb{P}^k of all polynomials of degree no greater than k on \mathbb{R}^n . For a finite subset Φ of \mathcal{L}_k^1 , we say that Φ satisfies the Strang-Fix condition of order k if there is a finite linear combination ϕ of the functions of Φ and their shifts such that $\hat{\phi}(0) \neq 0$ and $\partial^\alpha \hat{\phi}(2\pi\nu) = 0$, $|\alpha| \leq k - 1$, $\nu \in \mathbb{Z}^n$ with $\nu \neq 0$.

Lemma 1 . *Let Φ be a finite subset of \mathcal{L}_k^∞ that has L^2 -stable shifts. Then Φ satisfies the Strang-Fix condition of order k if and only if $P_0 q = q$ for any $q \in \mathbb{P}^{k-1}$.*

Moreover, if this is the case, then we have $\|P_l f - f\|_p \leq C \lambda_0^{-lk} \sum_{|\alpha|=k} \|\partial^\alpha f\|_p$ for any f in the Sobolev space $W_k^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$), with a constant C independent of f, p and l where λ_0 is the least value of absolute values of eigenvalues of the dilation matrix M , that is, $W_k^p(\mathbb{R}^n) \subset B_{pq}^{s,\lambda_0}(\Phi)$ if $0 < s < k$ and $1 \leq q \leq \infty$.

Proof. We can prove by the same way of [8, Theorem 5.2]. We will omit its details.

4 Characterization of Besov spaces

Let Π be a self-affine lattice tiling $\{T + \nu\}_{\nu \in \mathbb{Z}^n}$ and Π_l denote the subdivision $\{M^{-l}(T + \nu)\}_{\nu \in \mathbb{Z}^n}$ of \mathbb{R}^n for a nonnegative integer l . Let $\Phi = \{\phi_1, \dots, \phi_N\}$ be a finite subset of \mathcal{L}^∞ and λ_0 the least value of absolute values of eigenvalues of the dilation matrix M .

Proposition 3 . *Given $1 \leq p, q \leq \infty$ and $k > s > 0$. Assume that a finite subset $\Phi = \{\phi_1, \dots, \phi_N\}$ of \mathcal{L}_k^∞ satisfies*

- (a) Φ has L^2 -stable shifts,
- (b) Φ is M -refinable,
- (c) Φ satisfies the Strang-Fix condition of order k .

Then we have $B_{pq}^s(M) \subset B_{pq}^{s,\lambda_0}(\Phi)$.

Proof. We shall prove for any $f \in B_{pq}^s(M)$ by the same routine of the proof of Theorem 1,

$$\left(\sum_{l=0}^{\infty} (\lambda_0^{ls} \sigma_l^p(f))^q \right)^{1/q} \leq C \|f\|_{B_{pq}^s(M)}$$

where σ_l^p is given in (10) associated with Φ . We use the same notations in Theorem 1. We choose a function χ in $C_c^\infty(\mathbb{R}^n)$ such that $\int |\chi(u)| du = 1$ and $\text{supp } \chi \subset \{u \in \mathbb{R}^n : |u| < r/2k\}$ where r is the positive number given in (7). We write $\chi_l(u) = m^l \chi(M^l u)$, $h_l(x) = \int (f(x) - \Delta_u^k f(x)) \chi_l(u) du$ and $g_l = P_l h_l - h_l$ where P_l is given in (9) associated with Φ . Then we have for $1 \leq p \leq \infty$,

$$\|f - P_l f\|_p \leq \|f - h_l\|_p + \|g_l\|_p + \|P_l h_l - P_l f\|_p \leq C \|f - h_l\|_p + \|g_l\|_p \equiv C I_1 + I_2.$$

Obviously we have :

$$I_1 \leq C \sup_{k|M^l u| < r/2} \|\Delta_u^k f\|_p.$$

We shall give an estimate of I_2 by (1):

$$I_2 = \left(\sum_{Q \in \Pi_l} \int_Q |g_l(x)|^p dx \right)^{1/p} = \left(\sum_{\nu \in \mathbb{Z}^n} \int_{M^{-l} T} |g_l(x - M^{-l} \nu)|^p dx \right)^{1/p}. \quad (17)$$

Let q_z be the $(k-1)$ -th Taylor polynomial of h_l about $z \in \mathbb{R}^n$ and let r_z be the corresponding remainder. Since Φ satisfies the Strang-Fix condition of order k , we see from Lemma 1

$$g_l(x - M^{-l} \nu) = P_l r_{x - M^{-l} \nu}(x - M^{-l} \nu) = m^l \int K(M^l x, M^l y) r_{x - M^{-l} \nu}(y - M^{-l} \nu) dy$$

where $K(x, y) = \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} \phi_j(x - \nu) \bar{\phi}_j(y - \nu)$.

To estimate I_2 , we use

$$r_{x - M^{-l} \nu}(y - M^{-l} \nu) = \int_0^1 \sum_{|\beta|=k} \frac{k}{\beta!} \partial^\beta h_l(x + t(y - x) - M^{-l} \nu) (1-t)^{k-1} (y - x)^\beta dt,$$

and

$$\begin{aligned} |\partial^\beta h_l(x)| &\leq C \sum_{e=1}^k \left(\int_{|u| < r/2k} |f(x - e M^{-l} u)|^p du \right)^{1/p} \\ &\leq C \sum_{e=1}^k (m^l \int_{|M^l u| < r e/2k} |f(x - u)|^p du)^{1/p} \leq C m^{l/p} \left(\int_{|M^l u| < r/2} |f(x - u)|^p du \right)^{1/p}. \end{aligned}$$

Hence we get an estimate:

$$\begin{aligned} &\left(\sum_{\nu \in \mathbb{Z}^n} |r_{x - M^{-l} \nu}(y - M^{-l} \nu)|^p \right)^{1/p} \\ &\leq C \int_0^1 \sum_{|\beta|=k} \left(\sum_{\nu} |\partial^\beta h_l(x + t(y - x) - M^{-l} \nu)|^p \right)^{1/p} (1-t)^{k-1} |x - y|^k dt \\ &\leq C \int_0^1 \sum_{|\beta|=k} \left(\sum_{\nu} m^l \int_{|M^l u| < r/2} |f(x + t(y - x) - M^{-l} \nu - u)|^p du \right)^{1/p} (1-t)^{k-1} |x - y|^k dt \\ &\leq C \int_0^1 m^{l/p} \left(\sum_{\nu} \int_{M^{-l}(T+\nu)} |f(x + t(y - x) + u)|^p du \right)^{1/p} (1-t)^{k-1} |x - y|^k dt \\ &\leq C \int_0^1 m^{l/p} \|f\|_p (1-t)^{k-1} |x - y|^k dt \leq C |x - y|^k m^{l/p} \|f\|_p. \end{aligned}$$

Hence, since $\Phi \in \mathcal{L}_k^\infty$, we get an estimate of I_2 in (17):

$$\begin{aligned}
I_2 &\leq C m^l \left(\int_{M^{-l}T} \sum_{\nu} \left(\int |K(M^l x, M^l y)| |r_{x-M^{-l}\nu}(y - M^{-l}\nu)| dy \right)^p dx \right)^{1/p} \\
&\leq C m^l \left(\int_{M^{-l}T} \left(\int |K(M^l x, M^l y)| \left(\sum_{\nu} |r_{x-M^{-l}\nu}(y - M^{-l}\nu)|^p \right)^{1/p} dy \right)^p dx \right)^{1/p} \\
&\leq C m^{l+1/p} \|f\|_p \left(\int_{M^{-l}T} \left(\int |K(M^l x, M^l y)| |x - y|^k dy \right)^p dx \right)^{1/p} \\
&\leq C \|f\|_p \left(\int_T \left(\int |K(x, y)| |M^{-l}(x - y)|^k dy \right)^p dx \right)^{1/p} \\
&\leq C \|f\|_p \lambda_0^{-lk} \left(\int_T \left(\int |K(x, y)| |x - y|^k dy \right)^p dx \right)^{1/p} \leq C \|f\|_p \lambda_0^{-lk}.
\end{aligned}$$

Now we combine the estimates of I_1 and I_2 to write

$$\|f - P_l f\|_p \leq C I_1 + I_2 \leq C \left(\sup_{k|M^l u| < r/2} \|\Delta_u^k f\|_p + \lambda_0^{-lk} \|f\|_p \right).$$

This implies that

$$\left(\sum_{l=0}^{\infty} (\lambda_0^{ls} \sigma_l^p(f))^q \right)^{1/q} \leq C \|f\|_{B_{pq}^s(M)}.$$

This completes the proof of Proposition 3.

Even if Φ in Proposition 3 is M -refinable for a.e. $x \in \mathbb{R}^n$, Proposition 3 is true. Hence we have a following corollary:

Corollary. *Given $1 \leq p, q \leq \infty$ and $0 < s < k$. Then*

$$B_{pq}^s(M) \subset B_{pq}^{s, \lambda_0}(\{p_\alpha\}_{|\alpha| < k})$$

where the functions p_α are given in (3).

Remark 3. (a) We can define the operator $P_{\Pi_l}(l = 0, 1, 2, \dots)$ associated with $\Phi = \{p_\alpha\}_{|\alpha| \leq k}$, given in (9). Then we have $P_{\Pi_l} f(x) = \sum_{Q \in \Pi_l} P_Q f(x)$ where

$$P_Q f = \sum_{|\alpha| \leq k} \langle f, \tilde{p}_\alpha(M^l \cdot - x_0) \rangle m^l p_\alpha(M^l x - x_0)$$

for $Q = M^{-l}(T + x_0)$ is of type . We denote by $\tilde{\omega}_p^k(f, \Pi_l)$ the error of L^p -approximation in (10) associated with $\Phi = \{p_\alpha\}_{|\alpha| \leq k}$. We can prove a following equivalences for $f \in L^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$) by using the results in [2] and [7]:

$$\tilde{\omega}_p^k(f, \Pi_l) \sim \left(\sum_{Q \in \Pi_l} \int_Q |f - P_Q f|^p dy \right)^{1/p} \sim \left(\sum_{Q \in \Pi_l} \inf_{P \in \mathbb{P}^k} \int_Q |f - P|^p dy \right)^{1/p}.$$

(b) Let $\Pi_l(x_0)$ denote $\{M^{-l}(T + \gamma) + x_0\}_{\gamma \in \mathbb{Z}^n}$ for $x_0 \in \mathbb{R}^n$. We write

$$\tilde{\omega}_p^k(f, \Pi_l(x_0)) = \left(\sum_{Q \in \Pi_l(x_0)} \int_Q |f(y) - P_Q f(y)|^p dy \right)^{1/p}$$

and $\tilde{\omega}_p^k(f, l) = \sup_{x_0 \in \mathbb{R}^n} \tilde{\omega}_p^k(f, \Pi_l(x_0))$. When the self-affine lattice tiling Π is the net of closed cubes generated by $T = [0, 1]^n$ and the dilation matrix M is $2Id$, we see for $1 \leq p, q \leq \infty$ and $k + 1 > s > 0$

$$\|f\|_{B_{pq}^s(\mathbb{R}^n)} \sim \|f\|_p + \left(\sum_{l=0}^{\infty} (\lambda_0^{ls} \tilde{\omega}_p^k(f, l))^q \right)^{1/q}.$$

(See [7]).

Theorem 3 . *Given $1 \leq p, q \leq \infty$ and $k > s > 0$. Assume that a finite subset $\Phi = \{\phi_1, \dots, \phi_N\}$ of \mathcal{L}_k^∞ satisfies*

- (a) Φ has L^2 -stable shifts,
- (b) Φ is M -refinable,
- (c) *there exists a positive number s_0 with $s_0 > s$ such that $\sup_{l \geq 0} \lambda_0^{ls_0} |\text{osc}_p^{k-1} \phi_j(\cdot, l)|_p < \infty$ for all $j = 1, \dots, N$,*
- (d) Φ satisfies the Strang-Fix condition of order k .

Then we have $B_{pq}^s(M) = B_{pq}^{s, \lambda_0}(\Phi)$ with equivalent norms

$$\|f\|_{B_{pq}^s(M)} \sim \|f\|_{B_{pq}^{s, \lambda_0}(\Phi)}$$

where the norms $\|f\|_{B_{pq}^s(M)}$ and $\|f\|_{B_{pq}^{s, \lambda_0}(\Phi)}$ are given in (6) and (11) respectively, and λ_0 is the least value of absolute values of eigenvalues of the dilation matrix M .

Remark 4. When $\{\phi_j\}_{j=1}^N$ have compact supports, we see that the condition (c) in Theorem 3 can be rephrased as :

- (c)' There exists a positive number $s_0 > s$ such that $\sup_{l \geq 0} \lambda_0^{ls_0} \|\text{osc}_p^{k-1} \phi_j(\cdot, l)\|_p < \infty$, (that is, $\phi_j \in B_{p\infty}^{s_0}(M)$ if $s_0 < k$) for all $j = 1, \dots, N$.

Proof of Theorem 3. This result is an immediate consequence of Proposition 2 and Proposition 3.

We say that a function on \mathbb{R}^n is k -regular if it is of class C^k and rapidly decreasing in the sense that $|\partial^\alpha f(x)| \leq C_N(1 + |x|)^{-N}$ for all $N = 0, 1, 2, \dots$ and all $|\alpha| \leq k$. Any k -regular function belongs to \mathcal{L}_N^∞ for any $N \geq 0$ and any k -regular function f satisfies the condition (c) in Theorem 3 : $\sup_{l \geq 0} \lambda_0^{lk} |\text{osc}_p^{k-1} f(\cdot, l)|_p < \infty$.

Corollary 1 . *Suppose that a dilation matrix is of the form $M = \lambda_0 Id$ with $\lambda_0 > 1$. Let $1 \leq p, q \leq \infty$ and $k > s > 0$. Assume that a finite subset $\Phi = \{\phi_1, \dots, \phi_N\}$ of k -regular functions on \mathbb{R}^n satisfies:*

- (a) Φ has L^2 -stable shifts,
- (b) Φ is M -refinable.

Then there exists a set $\{\psi_j^\epsilon\}_{j=1, \dots, N}^{\epsilon=1, \dots, m-1}$ of k -regular wavelets associated with Φ , and we have equivalence of norms, if one of them exist, for wavelet expansion given in (15) with the usual modification for $q = \infty$:

$$\begin{aligned} & \text{(i)} \quad \|f\|_{B_{pq}^s(M)}, \\ \sim & \text{(ii)} \quad \|f\|_{B_{pq}^{s, \lambda_0}(\Phi)}, \\ \sim & \text{(iii)} \quad \sum_{j=1}^N \|a_{j0}\|_{l^p} + \left(\sum_{l=0}^{\infty} (\lambda_0^{l(s+n/2-n/p)}) \sum_{j=1}^N \sum_{\epsilon=1}^{m-1} \|b_{jl}^\epsilon\|_{l^p}^q \right)^{1/q}. \end{aligned}$$

Proof. From [15, Theorem 5.15], for a finite subset Φ of k -regular functions there exists an associated set of k -regular wavelets for a general dilation matrix M if $m > (n+1)/2$. Since a finite subset of k -regular functions satisfies the Strang-Fix condition of order $k+1$ in the case $M = \lambda_0 Id$ (See [9, Theorem 4 in 2.6] and Lemma 1), we have the equivalence of (i) and (ii) from Theorem 3. The equivalence of (ii) and (iii) can be proved by Theorem 2.

We define the tensor product B-spline by $\mathcal{M}_k = \prod_{i=1}^n M_k(x_i)$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $k = 1, 2, \dots$ where $M_k(t)$ is the k -th order central B-spline, that is, $\hat{M}_k(t) = (\frac{\sin(t/2)}{t/2})^k$. Let us denote by $\{e^i\}_{i=1}^n$ the set of unit vectors in \mathbb{R}^n . We put $e^{n+1} = \sum_{i=1}^n e^i$, and $X = \{x^1, \dots, x^{d_0}\}$ with $x^1 = e^1, \dots, x^{d_1} = e^1, x^{d_1+1} = e^2, \dots, x^{d_1+d_2} = e^2, \dots, x^{d_1+\dots+d_n+1} = e^{n+1}, \dots, x^{d_0} = e^{n+1}$ where $d_0 = d_1 + \dots + d_{n+1}$. We denote the box spline $B(x, X)$ corresponding to X given by $\hat{B}(x, X) = (2\pi)^{-n/2} \prod_{j=1}^{d_0} \frac{1 - e^{ix^j \cdot x}}{ix^j \cdot x}$. In the case that the self-affine lattice tiling is the net of closed cubes generated by $T = [0, 1]^n$ and the dilation matrix is $2Id$, the k -th order tensor product B-spline \mathcal{M}_k satisfies the conditions of Theorem 3, particularly, $\mathcal{M}_k \in B_{p\infty}^{k-1+1/p}(\mathbb{R}^n)$ and \mathcal{M}_k satisfies the Strang-Fix condition of order k . The above box spline $B(x, X)$ also satisfies the conditions of Theorem 3 replacing the above k by $k = \min\{d_i + d_j : i, j = 1, \dots, n+1, i \neq j\}$. Hence we get results of [3] and [12].

Corollary 2 . *Suppose that the self-affine lattice tiling is the net $\Pi = \{T + \nu\}_{\nu \in \mathbb{Z}^n}$ of closed cubes generated by $T = [0, 1]^n$ and the dilation matrix is $2Id$. Then Theorem 3 remains true for the tensor product B-spline $\Phi = \{\mathcal{M}_k\}$ or the box spline $\Phi = \{B(x, X)\}$.*

More general results have been given in W.Sickel [12] and R.A. DeVore, B. Jarwerth and V. Popov [4] in a case of some compactly supported functions. Theorem 3 is a generalization of those results.

A following proposition is a pointwise version of Corollary 1 in Theorem 3.

Proposition 4 . *Suppose that a dilation matrix is of the form $M = \lambda_0 Id$ with $\lambda_0 > 1$ and $k > s > 0$. Assume that a finite subset $\Phi = \{\phi_1, \dots, \phi_N\}$ of k -regular functions on \mathbb{R}^n satisfies:*

- (a) Φ has L^2 -stable shifts,
- (b) Φ is M -refinable.

Then for $x \in \mathbb{R}^n$ and a bounded function f on \mathbb{R}^n , following properties are equivalent:

- (i) $f \in C^s(x)$,
- (ii) $|f(y) - P_l f(y)| \leq C(\lambda_0^{-l} + |x - y|)^s \quad l \geq 0$

where $P_l f$ is given in (9).

Proof. This can be proved by the same way as in Proposition 1. See [1, Theorem 3].

Corollary. *Suppose that the conditions in Proposition 4 are satisfied. Let $s > s' > 0$.*

- (a) *If $f \in C^s(x)$, we have*

$$|R_l f(y)| \leq C(\lambda_0^{-l} + |x - y|)^s \quad l = 0, 1, 2, 3, \dots$$

where $R_l f$ is given in (12).

If it holds

$$|R_l f(y)| \leq C \lambda_0^{-sl} (1 + \lambda_0^l |x - y|)^{s'} \quad l = 0, 1, 2, 3, \dots,$$

then $f \in C^s(x)$.

(b) If $f \in C^s(x)$, we have

$$|b_{jl}^\epsilon(\nu)| \leq C \lambda_0^{-(s+\frac{n}{2})l} (1 + |\lambda_0^l x - \nu|)^s$$

for $j = 1, \dots, N, l = 1, 2, 3, \dots, \epsilon = 1, \dots, m-1$ and any $\nu \in \mathbb{Z}^n$ where $b_{jl}^\epsilon(\nu)$ is given in (16).

If it holds

$$|b_{jl}^\epsilon(\nu)| \leq C \lambda_0^{-(s+\frac{n}{2})l} (1 + |\lambda_0^l x - \nu|)^{s'} \quad \text{for } j = 1, \dots, N, \quad l = 1, 2, 3, \dots \text{ and } \epsilon = 1, \dots, m-1$$

and any $\nu \in \mathbb{Z}^n$, then $f \in C^s(x)$.

(c) For $\{a_{jl}(\nu)\}$ given in (13), if it holds

$$|a_{jl}(\nu)| \leq C \lambda_0^{-sl} (1 + |\lambda_0^l x - \nu|)^{s'} \quad j = 1, \dots, N, \quad l > 0 \text{ and } \nu \in \mathbb{Z}^n,$$

then $f \in C^s(x)$.

5 Scaling exponents

For $1 \leq p, q \leq \infty$ we define $\alpha_{pq}(f) = \sup\{s \geq 0 : f \in B_{pq}^s(M)\}$ for functions $f \in L^p(\mathbb{R}^n)$. If there is not a positive number s with $f \in B_{pq}^s(M)$, then we define $\alpha_{pq}(f) = 0$. We remark that $\alpha_{pq}(f) > 0$ for any $f \in L^p(\mathbb{R}^n)$ in the case $1 \leq p < \infty$. In the same manner we define $\alpha_{pq}(f, x) = \sup\{s \geq 0 : f \in T_{pq}^s(x)\}$ for $x \in \mathbb{R}^n$ and bounded functions f on \mathbb{R}^n . We put $\alpha_p(f) = \alpha_{p\infty}(f)$, $\alpha(f) = \alpha_\infty(f)$, $\alpha_p(f, x) = \alpha_{p\infty}(f, x)$ and $\alpha(f, x) = \alpha_\infty(f, x)$.

We can prove a following proposition by the embedding theorem (See Remark 1, 2 and [11]).

Proposition 5 .

- (i) $\alpha_p(f) = \alpha_{p\eta}(f)$ for $1 \leq p, \eta \leq \infty$,
- (ii) $\alpha(f) > \alpha_p(f) - \frac{n}{p} \geq \alpha_q(f) - \frac{n}{q}$ for $1 \leq q \leq p < \infty$ when $M = \lambda_0 Id$,
- (iii) $\alpha_p(f, x) = \alpha_{p\eta}(f, x)$ for $1 \leq p, \eta \leq \infty$,
- (iv) $\alpha(f) \leq \alpha(f, x) \leq \alpha_p(f, x) \leq \alpha_q(f, x)$ for $1 \leq q \leq p < \infty$.

For $1 \leq p \leq \infty$ we have by Theorem 1 and Theorem B

$$\alpha_p(f) = -\frac{\log A_p(f)}{\log \lambda_0}$$

if the right hand side of the above equality is less than $k+1$ where

$$A_p(f) = \limsup_{l \rightarrow \infty} \|\text{osc}_p^k f(\cdot, l)\|_p^{1/l} = \limsup_{l \rightarrow \infty} \sup_{(k+1)|M^l u| < r/2} \|\Delta_u^{k+1} f\|_p^{1/l}$$

and furthermore when $M = \lambda_0 Id$ with $\lambda_0 > 1$

$$A_p(f) = \limsup_{l \rightarrow \infty} \|f - S_l f\|_p^{1/l} = \limsup_{l \rightarrow \infty} \|f_l\|_p^{1/l}.$$

For $1 \leq p \leq \infty$ we have by the corollary of Theorem 1

$$\alpha_p(f, x) = -\frac{\log A_p(f, x)}{\log \lambda_0}$$

if the right hand side of the above equality is less than $k + 1$ where

$$A_p(f, x) = \limsup_{l \rightarrow \infty} \text{osc}_p^k f(x, l)^{1/l} = \limsup_{l \rightarrow \infty} \sup_{(k+1)|M^l u| < r/2} \left(\frac{1}{|Q_l(x)|} \int_{Q_l(x)} |\Delta_u^{k+1} f(y)|^p dy \right)^{1/pl}.$$

Furthermore when $M = \lambda_0 Id$ with $\lambda_0 > 1$, we have by Proposition 1 and its corollary

$$\alpha(f, x) = \liminf_{\lambda_0^{-l} + |x-y| \rightarrow 0} \frac{\log |f(y) - S_l f(y)|}{\log(\lambda_0^{-l} + |x-y|)}$$

and, if $\alpha(f) > 0$

$$\alpha(f, x) = \liminf_{\lambda_0^{-l} + |x-y| \rightarrow 0} \frac{\log |f_l(y)|}{\log(\lambda_0^{-l} + |x-y|)}$$

where $S_l f$ and f_l are given for Littlewood-Paley decomposition in (8).

We can prove a following proposition by Theorem 2, Theorem 3, Proposition 4 and its corollary.

Proposition 6 . (i). Assume that a finite subset $\Phi = \{\phi_1, \dots, \phi_N\}$ of \mathcal{L}_k^∞ satisfies the conditions (a), (b), (c) and (d) of Theorem 3.

Then for $f \in L^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$) we have

$$\alpha_p(f) = -\frac{\log A_p(f)}{\log \lambda_0} = \frac{\log m}{p \log \lambda_0} - \frac{\log \rho_p(f)}{\log \lambda_0}$$

if the second and third parts of the above equality are less than $\min(k, s_0)$ where

$$A_p(f) = \limsup_{l \rightarrow \infty} \sigma_l^p(f)^{1/l} = \limsup_{l \rightarrow \infty} \|R_l(f)\|_p^{1/l}$$

and

$$\rho_p(f) = \limsup_{l \rightarrow \infty} \sum_{j=1}^N \|a_{jl}\|_{l^p}^{1/l} = \inf \limsup_{l \rightarrow \infty} \sum_{j=1}^N \|c_{jl}\|_{l^p}^{1/l}$$

and $\{a_{jl}\}$ is given by (13) and inf is taken over all admissible representations $f(x) = \sum_{j=1}^N \sum_{l=0}^\infty \sum_{\nu \in \mathbb{Z}^n} c_{jl}(\nu) \phi_j(M^l x - \nu)$ as in Theorem 2.

(ii). Furthermore when $m > (n+1)/2$, we have

$$\alpha_p(f) = (1/p - 1/2) \frac{\log m}{\log \lambda_0} - \frac{\log \rho'_p(f)}{\log \lambda_0}$$

if the right hand side of the above equality is less than $\min(k, s_0)$ where

$$\rho'_p(f) = \limsup_{l \rightarrow \infty} \sum_{j=1}^N \sum_{\epsilon=1}^{m-1} \|b_{jl}^\epsilon\|_{l^p}^{1/l}$$

and $\{b_{jl}^\epsilon\}$ is given in (16) for the wavelet expansion (15) associated with Φ .

(iii). Suppose that conditions in Proposition 4 hold for a bounded function f . Then we have

$$\alpha(f, x) = \liminf_{\lambda_0^{-l} + |x-y| \rightarrow 0} \frac{\log |f(y) - P_l f(y)|}{\log(\lambda_0^{-l} + |x-y|)}$$

if the right hand side of the above equality is less than k and,

$$\begin{aligned} \alpha(f, x) &= \liminf_{\lambda_0^{-l} + |x-y| \rightarrow 0} \frac{\log |R_l f(y)|}{\log(\lambda_0^{-l} + |x-y|)} \\ &= \liminf_{\lambda_0^{-l} + |x-\lambda_0^{-l}\nu| \rightarrow 0} \inf_j \frac{\log \lambda_0^{\frac{n}{2}l} |b_{jl}^\epsilon(\nu)|}{\log(\lambda_0^{-l} + |x-\lambda_0^{-l}\nu|)} \leq \liminf_{\lambda_0^{-l} + |x-\lambda_0^{-l}\nu| \rightarrow 0} \inf_j \frac{\log |a_{jl}(\nu)|}{\log(\lambda_0^{-l} + |x-\lambda_0^{-l}\nu|)} \end{aligned}$$

if $\alpha(f) > 0$ and the right hand side of the above inequality is less than k where $P_l f$, $R_l f$ and $\{a_{jl}\}$ are given in (9), (12) and (13) respectively.

Let $\Pi = \{T + \nu\}_{\nu \in \mathbb{Z}^n}$ be a self-affine lattice tiling with a dilation matrix M and a set Γ_0 of digits, and Π_l denote the subdivision $\{M^{-l}(T + \nu)\}_{\nu \in \mathbb{Z}^n}$ of \mathbb{R}^n for a nonnegative integer l . We write $Q = M^{-l}(T + \nu_Q)$ for $Q \in \Pi_l$. Let $\Pi_l(T) = \{Q \in \Pi_l : Q \subset T\}$ and $\Pi(T) = \cup_{l=0}^{\infty} \Pi_l(T)$. We put $\Gamma_0 = \{\gamma_1, \dots, \gamma_m\}$. Then from (1) for $Q \in \Pi_l(T)$, ν_Q is of a form $\nu_Q = M^{l-1}\gamma_{i_1} + \dots + \gamma_{i_l}$, $\gamma_{i_1}, \dots, \gamma_{i_l} \in \Gamma_0$ and let $M_Q y = M^l y - \nu_Q$ and $\mu_Q = \mu_{i_1} \dots \mu_{i_l}$ for $l > 0$ where $\mu_1, \mu_2, \dots, \mu_m$ are real or complex numbers with $0 < |\mu_i| < 1$, $i = 1, \dots, m$. For $l = 0$ we put $M_T = Id$ and $\mu_T = 1$.

From now we suppose that a dilation matrix M is of a form $M = \lambda_0 Id$ with $\lambda_0 > 1$ and we consider a bounded function f which is given by a series

$$f(y) = \sum_{Q \in \Pi(T)} \mu_Q \phi(M_Q y), \quad y \in \mathbb{R}^n \quad (18)$$

where a function ϕ is bounded and zero outside T^o . We remark that $\alpha(f) \leq \alpha(\phi)$. Let

$$\tau_0(x) \equiv \liminf_{l \rightarrow \infty} \inf_{K_l(x) \ni Q} \frac{\log |\mu_Q|}{\log(\lambda_0^{-l} + |x - \lambda_0^{-l}\nu_Q|)} = \liminf_{l \rightarrow \infty} \inf_{K_l(x) \ni Q} \frac{\log |\mu_Q|}{\log \lambda_0^{-l}}$$

where $K_l(x) \equiv \{Q \in \Pi_l(T) : B(x, \lambda_0^{-l}) \cap Q \neq \emptyset\}$ and $B(x, \lambda_0^{-l})$ is a ball centered at x with a radius λ_0^{-l} . When $x \in \Omega \equiv \cap_{l=0}^{\infty} \cup_{Q \in \Pi_l(T)} Q^o$ (the interior of Q) there exists a unique sequence $\{Q_{l,x}\}_{l \geq 0}$ such that $Q_{l,x} \in \Pi_l(T)$ and $x \in Q_{l,x}^o$. Then we have for $x \in \Omega$

$$\tau_0(x) = \liminf_{l \rightarrow \infty} \frac{\log |\mu_{Q_{l,x}}|}{\log \lambda_0^{-l}}.$$

Let for $x \in \Omega$

$$\tau_1(x) \equiv \liminf_{l \rightarrow \infty} \frac{\log |\mu_{Q_{l,x}}|}{\log \Delta_l(x)}$$

where $\Delta_l(x) = \text{dist}(x, \partial Q_{l,x})$ is the distance from x to the boundary $\partial Q_{l,x}$ of $Q_{l,x}$. We remark for $x \in \Omega$, $\tau_0(x) = \tau_1(x)$ if $\sup_{l \geq 0} \frac{\Delta_l(x)}{\Delta_{l+1}(x)} < \infty$.

A following theorem may be proved by the same way as in [11].

Theorem 4 . *Let f and ϕ be bounded functions given in (18). Then we have*

(i) $\alpha(f, x) \geq \min(\alpha(\phi), \tau_0(x))$ for $x \in T$,

(ii) $\alpha(f, x) \geq \min_i(\alpha(\phi, \Omega_i), \tau_1(x))$ for $x \in \Omega$ with $\sup_{l \geq 0} \frac{\Delta_l(x)}{\Delta_{l+1}(x)} < \infty$

where $\Omega_i \equiv M^{-1}(T^o + \gamma_i)$, $\gamma_i \in \Gamma_0$, $i = 1, \dots, m$ and $\alpha(\phi, \Omega_i) = \sup\{s \geq 0 : \phi \in C^s(\Omega_i)\}$ and $C^s(\Omega_i)$ is defined as the Besov space $B_{\infty\infty}^s(\Omega_i)$ on Ω_i .

(iii) Suppose that $\phi \in C^\infty(\Omega_i)$, $i = 1, \dots, m$ and there exist a positive number s_0 and $y_0 \in T^o$ such that

$$\sup_{l \geq 0} \sup_y \frac{|f_l(y)|}{(\lambda_0^{-l} + |y - y_0|)^{s_0}} = \infty.$$

Then $\tau_0(x) \geq \alpha(f, x)$ for $x \in T$.

The proof of Theorem 4 is not difficult. We will omit its details.

Corollary. *Let ϕ be a bounded function on \mathbb{R}^n such that $\phi \in C^\infty(\Omega_j)$, $j = 1, \dots, m$ and $\phi = 0$ outside T^o . Consider a bounded function f given by (18) satisfying the condition (iii) in Theorem 4. Then we have*

(i) $\tau_0(x) \geq \alpha(f, x) \geq \min(\alpha(\phi), \tau_0(x))$, $x \in T$,

(ii) for x in Ω with $\sup_{l \geq 0} \frac{\Delta_l(x)}{\Delta_{l+1}(x)} < \infty$, $\alpha(f, x) = \tau_0(x) = \tau_1(x)$.

Examples. We consider a self-affine tiling $\Pi = \{T + \nu\}_{\nu \in \mathbb{Z}}$ such that a tile $T = [0, 1]$ and a dilation $M = 2Id$ on \mathbb{R} .

(a) We consider the Takagi function such that

$$f(x) = \sum_{l=0}^{\infty} \sum_{Q \in \Pi_l(T)} \mu^l \phi(M_Q x), \quad \forall x \in \mathbb{R}$$

where $0 < \mu < 1$ and ϕ is a bounded function such that $\phi(x) = x$ ($0 < x \leq \frac{1}{2}$), $\phi(x) = 1 - x$ ($\frac{1}{2} \leq x < 1$), $\phi(x) = 0$ (otherwise). Let $\tau = \frac{\log \mu}{\log 2^{-1}}$. Then from the corollary of

Theorem 4, if $\tau \leq 1$, $\tau = \alpha(f, x)$ for each $x \in T$.

(b) We consider the Weierstrass function $f(x) = \sum_{l=0}^{\infty} \mu^l \phi(2^l x)$ with $0 < \mu < 1$ and $\phi(x) = \sin 2\pi x$ ($x \in \mathbb{R}$). The proof of Theorem 4 can be also applied to this function case. Then we have

$$\tau = \alpha(f, x), \quad \forall x \in \mathbb{R}.$$

where the constant $\tau = \frac{\log \mu}{\log 2^{-1}}$ is given in the part (a) above.

(c) We consider L  vy's function

$$f(x) = \sum_{l=0}^{\infty} \sum_{Q \in \Pi_l(T)} 2^{-l} \phi(M_Q x), \quad \forall x \in \mathbb{R}$$

where $\phi(x) = x - \frac{1}{2}$ ($0 < x < 1$), $\phi(x) = 0$ (otherwise). Then we can see that $1 = \tau_1(x) = \alpha(f, x)$ for a point x in Ω with $\sup_{l \geq 0} \frac{\Delta_l(x)}{\Delta_{l+1}(x)} < \infty$.

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